

A Parameter Varying Lyapunov Function approach for Tracking Control for Takagi-Sugeno Class of Nonlinear Systems

M. Ezzeldin, A. Jokic and P.P.J. van den Bosch

Abstract—A reference model tracking control technique for nonlinear systems based on the Takagi-Sugeno model is proposed. The control design synthesis is aimed to reduce the tracking error for all bounded reference inputs and disturbances and to guarantee \mathcal{L}_2 gain performance. A nonlinear static output feedback controller is proposed to tackle this problem. Unlike the approaches using a single quadratic Lyapunov function, a parameter varying quadratic Lyapunov function is employed in our approach. The controller synthesis is formulated in terms of a feasibility problem of a set of linear matrix inequalities, which can be efficiently solved. A simulation example of a two-link robot system demonstrates the tracking performance and the validity of the proposed approach.

I. INTRODUCTION

The Takagi-Sugeno (T-S) model-based approach has nowadays become popular since it showed its efficiency to control complex nonlinear systems and has been used for many applications.

The Takagi-Sugeno (T-S) models were introduced in [1], with the purpose of approximating complex nonlinear systems and can be described as the weighted sum of local linear models. The weights depend on the working point of the nonlinear system. The T-S models can be seen as a generalization of piece-wise linear systems. Based on the so called Universal Approximation theorem [2], appropriate weighting of the local models can reproduce the original nonlinear system arbitrary well. T-S models have led to the design of a feedback controller for each local model, which are further combined into a single overall controller. Stability and stabilization analyses, of the T-S model, have been investigated through Lyapunov direct method [3]-[5]. The key point of the proposed approaches is to achieve conditions in terms of LMIs (linear matrix inequalities). For different approaches on this subject the interested reader is referred to [6]-[10].

Tracking control design is an important issue for practical applications, for example, in robotic tracking control, missile tracking control and attitude tracking control of aircraft. However, there are very few studies concerning tracking control design based on the T-S model [11]-[14]. Most of the available tracking control techniques using T-S models

are either based on state or observer based output feedback and based on the conventional, single quadratic Lyapunov function. These techniques depend on finding a constant positive definite matrix of a quadratic Lyapunov function to satisfy the stability condition of the system. In [14] a fuzzy tracking controller design for discrete time systems is proposed using the feedback linearization technique. Similarly, [15] has established a simple necessary and sufficient condition to determine local stability for the fuzzy systems and has derived a fuzzy tracking controller. In [11], an H_∞ performance, related to the tracking error for bounded reference inputs, is formulated and an observer-based fuzzy controller is developed.

In this paper, we consider the tracking problem for nonlinear systems described by a T-S model with external disturbances. A reference model tracking based static output feedback controller is proposed to achieve minimal tracking error in terms of the desired \mathcal{L}_2 gain tracking performance. Instead of using the conventional quadratic Lyapunov function, in our approach a parameter varying quadratic Lyapunov function is employed. The parameter varying Lyapunov function approach for stabilization problem was already studied in [7], where the considered Lyapunov function is a weighted sum of several quadratic Lyapunov functions. The weights are the same ones as used in the T-S model. Note that this Lyapunov function, which is used in our approach as well, presents a significantly wider class of functions than the conventional quadratic one. Moreover, this Lyapunov function is smooth, unlike the piecewise Lyapunov function. Therefore, the so-called boundary condition problem is not present for the proposed framework. Initial formulation of the considered \mathcal{L}_2 gain tracking problem is necessarily given in terms of a nonlinear matrix inequality. A transformation is proposed to transform this nonlinear inequality into a set of LMIs for which efficient optimization techniques are available. Furthermore, the stability of the proposed controller is guaranteed for a general class of T-S systems. The main contributions of this paper are summarized as follows. The \mathcal{L}_2 gain static output feedback control design for the model tracking control is extended from linear system toward nonlinear system using nonlinear control. Moreover, based on the proposed approach, a simple algorithm in terms of LMI optimization technique is developed.

The remainder of the paper is organized as follows. Section II presents basic definitions of the T-S model and the problem formulation, while Section III introduces \mathcal{L}_2 gain based tracking control. Section IV presents the LMI formulation

This work has been carried out as part of the OCTOPUS project with Océ Technologies B.V. under the responsibility of the Embedded Systems Institute. This project is partially supported by the Netherlands Ministry of Economic Affairs under the Bsik program.

M. Ezzeldin, A. Jokic and P.P.J. van den Bosch are with the Department of Electrical Engineering, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands. m.ezz@tue.nl, a.jokic@tue.nl and p.p.j.v.d.bosch@tue.nl

of the proposed approach. A simulation example is given in section V to illustrate the design effectiveness. Concluding remarks are collected in section VI.

II. PROBLEM FORMULATION

In this section, we recall the definition and some properties of the T-S model which is used to approximate a class of nonlinear system in the form:

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + g(x(t))u(t) + B_w w(t), \\ y(t) &= Cx(t), \end{aligned} \quad (1)$$

where $x(t) \in \mathbb{R}^n$ denotes the state vector, $u(t) \in \mathbb{R}^m$ denotes the control input, $w(t) \in \mathbb{R}^{m_w}$ denotes the unknown but bounded disturbance, the vector fields $f(x(t))$ and $g(x(t))$ are smooth functions with $f(0) = 0$, and $y(t) \in \mathbb{R}^p$ is the output of the system which depends linearly on the states with $C \in \mathbb{R}^{p \times n}$.

Next, we will formally present the T-S model as a weighed average of several linear models through weighting functions. First, consider a set of linear models of the form:

$$\begin{aligned} \dot{x}(t) &= A_i x(t) + B_i u(t) + B_{wi} w(t), \\ y(t) &= C_i x(t), \end{aligned} \quad (2)$$

for $i = 1, \dots, L$, where $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times m}$, $C_i \in \mathbb{R}^{p \times n}$, and L is the number of linear models. The T-S model is defined as:

$$\Sigma : \begin{cases} \dot{x}(t) = \sum_{i=1}^L h_i(z(t)) (A_i x(t) + B_i u(t) + B_{wi} w(t)), \\ y(t) = \sum_{i=1}^L h_i(z(t)) C_i x(t), \end{cases} \quad (3)$$

with $h_i(z(t)) \geq 0$ for $i = 1, \dots, L$, $\sum_{i=1}^L h_i(z(t)) = 1$ and $z(t)$ are triggering variables, which depend on the current value of system states $x(t)$. In this paper, we will assume that $C_i = C$ for $i = 1, \dots, L$, what is in conformance with the output of the original nonlinear system (1).

Remark 1: An example for the membership functions $h_i(z(t))$ is shown in Fig. 1. Since in this paper we are concerned with the output feedback control, the triggering variables $z(t)$ are considered to be dependent only on the

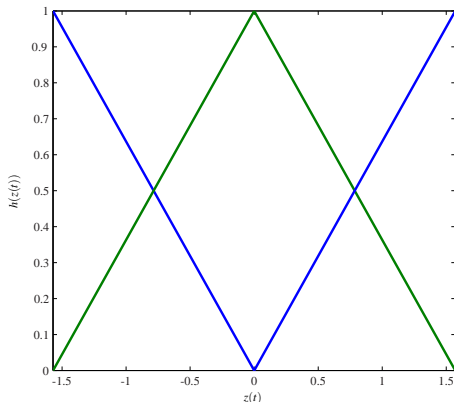


Fig. 1. Normalized membership functions.

measured output $y(t)$.

The T-S model (3) is a general nonlinear time-varying dynamic system. Without going into more details, the T-S model (3) can capture the piecewise linear system (PWL) accurately if $h_i(z(t))$ belongs to the discrete set $\{0, 1\}$.

To define the output tracking problem, we first introduce a linear time invariant reference model given by:

$$\begin{aligned} \dot{x}_r &= A_r x_r(t) + B_r r(t), \\ y_r &= C_r x_r(t), \end{aligned} \quad (4)$$

where $x_r(t)$ is the reference state, A_r is an asymptotically stable matrix, $r(t)$ is a bounded reference input and $y_r(t)$ is the reference output. It is assumed that, for all $t \geq 0$, $y_r(t)$ represents a desired trajectory for $y(t)$ to follow. The choice of the reference model parameters mainly depends on the desired closed loop behavior, e.g. system overshoot, settling time, etc. For more details about how to choose the reference model, see [16].

III. OPTIMIZED \mathcal{L}_2 GAIN BASED TRACKING CONTROL

In this paper, we are concerned with the controller synthesis which incorporates the reference model tracking performance criteria. Next, we recall the definition of dissipativity and the dissipativity characterization of the upper bounds for the \mathcal{L}_2 gain.

Definition 1: The system Σ with state $x \in \mathbb{R}^n$, input $w \in \mathbb{R}^{m_w}$ and output $y \in \mathbb{R}^p$, is dissipative with respect to the supply function $s : \mathbb{R}^{m_w} \times \mathbb{R}^p \rightarrow \mathbb{R}$ if there exists a positive definite continuous function $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$, called a storage function, such that $V(0) = 0$ and $V(x(t_1)) - V(x(t_0)) \leq \int_{t_0}^{t_1} s(w(t), y(t)) dt$ for all $t_1 \geq t_0$, and for all corresponding solutions (w, x, y) to the system Σ on the interval $[t_0, t_1]$.

Proposition 1: The system Σ has a finite \mathcal{L}_2 -gain from input w to the output y smaller than or equal to ρ if it is dissipative with respect to the supply function $s(w, y) = \rho^2 w^\top w - y^\top y$.

Next we present the structure of the output feedback controller considered in this paper. Consider a set of static feedback controllers of the following form:

$$u(t) = K_j [y(t) - y_r(t)] \quad \text{for } j = 1, \dots, L.$$

Then the overall controller is defined by:

$$u(t) = \sum_{j=1}^L h_j(z(t)) K_j [y(t) - y_r(t)], \quad (5)$$

where K_j are the controller gains for $j = 1, \dots, L$ and the functions $h_j(z(t))$, for $j = 1, \dots, L$ are the same ones as in (3).

The closed loop system can be expressed as the following

augmented system:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_r(t) \end{bmatrix} = \sum_{i=1}^L \sum_{j=1}^L h_i(z(t))h_j(z(t)) \left(\begin{bmatrix} A_i + B_i K_i C & -B_i K_j C_r \\ 0 & A_r \end{bmatrix} \begin{bmatrix} x(t) \\ x_r(t) \end{bmatrix} + \begin{bmatrix} B_{wi} & 0 \\ 0 & B_r \end{bmatrix} \begin{bmatrix} w(t) \\ r(t) \end{bmatrix} \right) \quad (6)$$

In the sequel, the following abbreviations are adopted:

$\tilde{x} := [x(t)^\top, x_r(t)^\top]^\top$, $\tilde{w} := [w(t)^\top, r(t)^\top]^\top$, $\tilde{C} := [C \ -C_r]$, $\tilde{A}_{ij} := \begin{bmatrix} A_i + B_i K_j C & -B_i K_j C_r \\ 0 & A_r \end{bmatrix}$, $E_i := \begin{bmatrix} B_{wi} & 0 \\ 0 & B_r \end{bmatrix}$, and the output tracking error $\tilde{y}(t) := y(t) - y_r(t)$. Therefore the closed loop system (6) can be expressed in the following compact form:

$$\begin{aligned} \dot{\tilde{x}} &= \sum_{i=1}^L \sum_{j=1}^L h_i h_j (\tilde{A}_{ij} \tilde{x} + E_i \tilde{w}), \\ \tilde{y} &= \tilde{C} \tilde{x}. \end{aligned} \quad (7)$$

For a given $\rho \in \mathbb{R}$, $\rho > 0$, the objective is to design a static feedback controller of the form (5) such that the corresponding closed loop system (7) has \mathcal{L}_2 gain from \tilde{w} to \tilde{y} less than or equal to ρ . The \mathcal{L}_2 gain tracking performance criteria for the closed loop system is defined in Proposition 1 with a supply function selected as follows:

$$s(\tilde{w}, \tilde{y}) = \rho^2 \tilde{w}^\top \tilde{w} - \tilde{y}^\top \tilde{y} = \rho^2 \tilde{w}^\top \tilde{w} - \tilde{x}^\top \tilde{Q} \tilde{x}, \quad (8)$$

$$\text{where } \tilde{Q} = \begin{bmatrix} C^\top C & -C^\top C_r \\ -C_r^\top C & C_r^\top C_r \end{bmatrix}.$$

Theorem 1: Suppose that $|\dot{h}_k| \leq \lambda_k$, $\lambda_k > 0$. Given the closed loop system (7), if there exist symmetric positive definite matrices P_k as a solution of the following matrix inequalities :

$$\tilde{A}_{ij}^\top P_k + P_k \tilde{A}_{ij} + \hat{P}_\lambda + \frac{1}{\rho^2} P_k E_i E_i^\top P_k + \tilde{Q} < 0, \quad i, j, k = 1, \dots, L \quad (9)$$

with $\hat{P}_\lambda = \sum_{k=1}^L \lambda_k (P_k + Y)$, and where λ_k are scalars, Y is a symmetric matrix, then the \mathcal{L}_2 gain tracking control performance in (8) is guaranteed for a prescribed value of ρ .

Proof: Consider the candidate Lyapunov function:

$$V(\tilde{x}(t)) = \sum_{k=1}^L h_k(z(t)) \tilde{x}^\top(t) P_k \tilde{x}(t), \quad (10)$$

where the function $h_k(z(t))$, for $k = 1, \dots, L$, are the same ones as in (3).

The time derivative of $V(t)$ along the systems trajectories is given by:

$$\begin{aligned} \dot{V}(\tilde{x}(t)) &= \sum_{k=1}^L h_k \dot{\tilde{x}}^\top(t) P_k \tilde{x}(t) + \sum_{k=1}^L h_k \tilde{x}^\top(t) P_k \dot{\tilde{x}}(t) \\ &\quad + \sum_{k=1}^L \dot{h}_k \tilde{x}^\top(t) P_k \tilde{x}(t). \end{aligned}$$

Furthermore, using (7) we have

$$\begin{aligned} \dot{V}(\tilde{x}(t)) &= \sum_{k=1}^L \sum_{i=1}^L \sum_{j=1}^L h_i h_j h_k (\tilde{x}^\top(t) \tilde{A}_{ij}^\top P_k \tilde{x}(t) + \tilde{x}^\top(t) P_k \tilde{A}_{ij} \tilde{x}(t) \\ &\quad + \tilde{w}^\top(t) E_i^\top P_k \tilde{x}(t) + \tilde{x}^\top(t) P_k E_i \tilde{w}(t)) + \sum_{k=1}^L \dot{h}_k \tilde{x}^\top(t) P_k \tilde{x}(t). \end{aligned} \quad (11)$$

Since $\sum_{k=1}^L h_k = 1$, hence, $\sum_{k=1}^L \dot{h}_k = 0$, it follows that $\sum_{k=1}^L \dot{h}_k Y = 0$ for arbitrary matrix Y . Adding $\sum_{k=1}^L \dot{h}_k Y$ to (11) with $Y \in \mathbb{R}^{2n \times 2n}$, and using the inequality $|\dot{h}_k| \leq \lambda_k$ it follows that:

$$\begin{aligned} \dot{V}(\tilde{x}(t)) &\leq \sum_{k=1}^L \sum_{i=1}^L \sum_{j=1}^L h_k h_i h_j \tilde{x}^\top(t) (\tilde{A}_{ij}^\top P_k + P_k \tilde{A}_{ij} \\ &\quad + \frac{1}{\rho^2} P_k E_i E_i^\top P_k) \tilde{x}(t) + \sum_{k=1}^L \lambda_k \tilde{x}^\top(t) (P_k + Y) \tilde{x}(t) + \rho^2 \tilde{w}^\top(t) \tilde{w}(t) \\ &\quad - \left(\frac{1}{\rho} E_i^\top P_k \tilde{x}(t) - \rho \tilde{w} \right)^\top \left(\frac{1}{\rho} P_k E_i \tilde{x}(t) - \rho \tilde{w} \right), \end{aligned} \quad (12)$$

inequality (12) further implies

$$\begin{aligned} \dot{V}(\tilde{x}(t)) &\leq \sum_{k=1}^L \sum_{i=1}^L \sum_{j=1}^L h_k h_i h_j \tilde{x}^\top(t) (\hat{P}_\lambda + \tilde{A}_{ij}^\top P_k + P_k \tilde{A}_{ij} \\ &\quad + \frac{1}{\rho^2} P_k E_i E_i^\top P_k) \tilde{x}(t) + \rho^2 \tilde{w}^\top(t) \tilde{w}(t). \end{aligned}$$

with the abbreviation $\hat{P}_\lambda := \sum_{k=1}^L \lambda_k (P_k + Y)$. Using (9) this further implies the following inequality

$$\dot{V}(\tilde{x}(t)) \leq \sum_{i=k}^L \sum_{i=1}^L \sum_{j=1}^L h_k h_i h_j \tilde{x}^\top(t) (-\tilde{Q}) \tilde{x}(t) + \rho^2 \tilde{w}^\top(t) \tilde{w}(t). \quad (13)$$

From the properties of $h_i(z(t))$, the inequality (13) implies the following inequality:

$$\dot{V}(\tilde{x}(t)) \leq \tilde{x}^\top(t) (-\tilde{Q}) \tilde{x}(t) + \rho^2 \tilde{w}^\top(t) \tilde{w}(t).$$

Integrating both sides from 0 to t_f we obtain,

$$\int_0^{t_f} \tilde{x}^\top(t) \tilde{Q} \tilde{x}(t) dt \leq \tilde{x}^\top(0) P \tilde{x}(0) + \rho^2 \int_0^{t_f} \tilde{w}^\top(t) \tilde{w}(t) dt$$

Therefore, according to Proposition 1 the \mathcal{L}_2 gain performance is achieved with a prescribed ρ . ■

To obtain a better tracking performance, the tracking control problem can be formulated as the following minimization problem:

$$\begin{aligned} \min_{P_k, K_j, \rho} \quad & \rho^2 \\ \text{subject to } & P_k > 0, \quad k = 1, \dots, L, \text{ and (9)} \end{aligned} \quad (14)$$

Theorem 2: Given the closed loop system (7). If there exist symmetric positive definite matrices P_k , $k = 1, \dots, L$, for the minimization problem (14), then the closed loop system (7) is quadratically stable.

Proof: Set $\tilde{w} = 0$ and consider the candidate Lyapunov function

$$V(\tilde{x}(t)) = \sum_{k=1}^L h_k \tilde{x}^\top(t) P_k \tilde{x}(t).$$

Then we can write the following equality

$$\begin{aligned}\dot{V}(\tilde{x}(t)) &= \sum_{k=1}^L h_k(\tilde{x}^\top(t)P_k\tilde{x}(t) + \tilde{x}^\top(t)P_k\dot{\tilde{x}}(t) + \dot{h}_k\tilde{x}^\top(t)P_k\tilde{x}(t)), \\ \dot{V}(\tilde{x}(t)) &= \sum_{k=1}^L \sum_{i=1}^L \sum_{j=1}^L h_k h_i h_j \tilde{x}^\top(t)(\tilde{A}_{ij}^\top P_k + P_k \tilde{A}_{ij} + \hat{P}_\lambda)\tilde{x}(t).\end{aligned}$$

Using (9), from the above inequality we deduce that $\dot{V}(\tilde{x}(t)) < 0$ ■

In the next section, we will present a constructive algorithm to compute the controller gains based on Theorem 1 in terms of an LMI feasibility problem.

IV. LMI FORMULATION

A class of numerical optimization problems, called linear matrix inequality (LMI) problems, has received significant attention over the last decade [17]-[19]. These optimization problems can be solved in polynomial time via the interior-point methods [18], and hence are tractable, at least in a theoretical sense.

Note that (9) is not an LMI in P_k and K_j as variables. Therefore, a transformation method is proposed to reformulate (9) as a feasibility problem of a set of LMIs. This transformation method is summarized in the following Lemma.

Lemma 1: The following LMI implies the inequality (9):

$$\begin{bmatrix} -P_k \tilde{Q} - \tilde{Q}^\top P_k + \hat{P}_\lambda + \tilde{Q} & (*) \\ \frac{1}{2\rho^2} E_i E_i^\top P_k + \tilde{A}_{ij} + \tilde{Q} + \frac{1}{\rho^2} P_k & -\frac{2}{\rho^2} I \end{bmatrix} < 0. \quad (15)$$

for $i, j, k = 1, \dots, L$

Proof: Pre- and Post-multiplying (15) by $[I \ P_k]$ and $[I \ P_k]^\top$ respectively, yields

$$\tilde{A}_{ij}^\top P_k + P_k \tilde{A}_{ij} + \hat{P}_\lambda + \frac{1}{\rho^2} P_k E_i E_i^\top P_k + \tilde{Q} < 0 \quad \blacksquare$$

Note that (15) is an LMI in P_k and K_j and can therefore be efficiently solved by a dedicated software.

According to the above analysis, the tracking control via nonlinear static output feedback is now summarized as follows:

Design Procedure:

- 1) Select weighing functions, $h_i(z(t))$ $i = 1, 2, \dots, L$, and construct a T-S model as in (3).
- 2) Select an initial value of ρ^2 and λ_k .
- 3) Solve the LMI problem in (15).
- 4) Decrease ρ^2 and repeat steps 3-4 until the LMI problem can not be solved.
- 5) Construct the controller (5) and verify if $|\dot{h}_k| \leq \lambda_k$.

Remark 2: The time derivative of (10) contains information on the time derivative of the weighting functions h_k . It's assumed that the time derivative of the weighting functions h_k are upper bounded with a given value, which may lead to some conservatism in the LMI problem. However, the

use of structural information on T-S model to construct the Lyapunov function is an advantage in comparison with disregarding the time derivative of the weighting functions. Moreover, the stability proof is based on this assumption, hence, it should be checked that this assumption is satisfied to have a complete stability proof. Furthermore, note that the number of LMIs (15) are L^3 which will have a high computational time, however, the number of the LMIs can be further reduced to $L^3 - 2L^2 + 2L$ taking into account that not all the T-S subsystems are triggered at the same time [5].

V. SIMULATION EXAMPLE

To illustrate the proposed design approach and show its efficiency, the results are applied to the two-link robot system. The dynamic equations of the two-link robot system are given as follows [11]:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = T \quad (16)$$

where

$$\begin{aligned}M(q) &= \begin{bmatrix} (m_1 + m_2)l_1^\top & m_2 l_1 l_2 (s_1 s_2 + c_1 c_2) \\ m_2 l_1 l_2 (s_1 s_2 + c_1 c_2) & m_2 l_2^2 \end{bmatrix} \\ C(q, \dot{q}) &= m_2 l_1 l_2 (c_1 s_2 - s_1 c_2) \begin{bmatrix} 0 & -\dot{q}_2 \\ -\dot{q}_1 & 0 \end{bmatrix} \\ G(q) &= \begin{bmatrix} -(m_1 + m_2)l_1 g s_4 \\ -m_2 l_2 g s_2 \end{bmatrix}\end{aligned}$$

and the system output $y = q = [q_1 \ q_2]^\top$, q_1, q_2 are generalized coordinates, $M(q)$ is the moment of inertia, $C(q, \dot{q})$ includes Coriolis, centrifugal forces and $G(q)$ is the gravitational force. Other quantities are: link mass m_1, m_2 [kg], link length l_1, l_2 [m], angular position q_1, q_2 [rad], applied torques $T = [T_1, T_2]^\top$ [Nm], the gravity $g = 9.8$ [m/s²] and $s_1 = \sin(q_1), s_2 = \sin(q_2), c_1 = \cos(q_1), c_2 = \cos(q_2)$. Let $x_1 = q_1, x_2 = \dot{q}_1, x_3 = q_2$ and $x_4 = \dot{q}_2$, then the Takagi-Sugeno model which approximates the nonlinear system (16) can be represented using the following set of linearized models [11]:

- R1 If $-\frac{\pi}{2} \leq x_1 \leq 0$ and $-\frac{\pi}{2} \leq x_3 \leq 0$
then $\dot{x} = A_1 x + B_1 u + w, z = C_1 x$
- R2 If $-\frac{\pi}{2} \leq x_1 \leq 0$ and $-\frac{\pi}{2} \leq x_3 \leq \frac{\pi}{2}$
then $\dot{x} = A_2 x + B_2 u + w, z = C_2 x$
- R3 If $-\frac{\pi}{2} \leq x_1 \leq 0$ and $0 \leq x_3 \leq \frac{\pi}{2}$
then $\dot{x} = A_3 x + B_3 u + w, z = C_3 x$
- R4 If $-\frac{\pi}{2} \leq x_1 \leq \frac{\pi}{2}$ and $-\frac{\pi}{2} \leq x_3 \leq 0$
then $\dot{x} = A_4 x + B_4 u + w, z = C_4 x$
- R5 If $-\frac{\pi}{2} \leq x_1 \leq \frac{\pi}{2}$ and $-\frac{\pi}{2} \leq x_3 \leq \frac{\pi}{2}$
then $\dot{x} = A_5 x + B_5 u + w, z = C_5 x$
- R6 If $-\frac{\pi}{2} \leq x_1 \leq \frac{\pi}{2}$ and $0 \leq x_3 \leq \frac{\pi}{2}$
then $\dot{x} = A_6 x + B_6 u + w, z = C_6 x$

R7 If $0 \leq x_1 \leq \frac{\pi}{2}$ and $-\frac{\pi}{2} \leq x_3 \leq 0$
then $\dot{x} = A_7x + B_7u + w$, $z = C_7x$

R8 If $0 \leq x_1 \leq \frac{\pi}{2}$ and $-\frac{\pi}{2} \leq x_3 \leq \frac{\pi}{2}$
then $\dot{x} = A_8x + B_8u + w$, $z = C_8x$

R9 If $0 \leq x_1 \leq \frac{\pi}{2}$ and $0 \leq x_3 \leq \frac{\pi}{2}$
then $\dot{x} = A_9x + B_9u + w$, $z = C_9x$

$$B_4 = \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, B_5 = \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \\ -1 & 2 \end{bmatrix}, B_6 = \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$B_7 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 1 & 2 \end{bmatrix}, B_8 = \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, B_9 = \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \\ -1 & 2 \end{bmatrix},$$

$$C_i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ for } i = 1, \dots, 9$$

where, $x = [x_1 \ x_2 \ x_3 \ x_4]^\top$, $u = [T_1 \ T_2]^\top$ and

$$w = [w_1 \ w_2 \ w_3 \ w_4]^\top.$$

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 5.927 & -0.001 & -0.315 & -8.4 \times 10^{-6} \\ 0 & 0 & 0 & 1 \\ -6.859 & 0.002 & 3.155 & 6.2 \times 10^{-6} \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3.0428 & -0.0011 & 0.1791 & -0.0002 \\ 0 & 0 & 0 & 1 \\ 3.5436 & 0.0313 & 2.5611 & 1.14 \times 10^{-5} \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 6.2728 & 0.003 & 0.4339 & -0.0001 \\ 0 & 0 & 0 & 1 \\ 9.1040 & 0.0158 & -1.0574 & -3.2 \times 10^{-5} \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 6.4535 & 0.0017 & 1.2427 & 0.0002 \\ 0 & 0 & 0 & 1 \\ -3.1873 & -0.306 & 5.1911 & -1.8 \times 10^{-5} \end{bmatrix}$$

$$A_5 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 11.1336 & 0 & -1.8145 & 0 \\ 0 & 0 & 0 & 1 \\ -9.0918 & 0 & 9.1638 & 0 \end{bmatrix}$$

$$A_6 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 6.1702 & -0.001 & 1.687 & -0.0002 \\ 0 & 0 & 0 & 1 \\ -2.3559 & 0.0314 & 4.5298 & 1.1 \times 10^{-5} \end{bmatrix}$$

$$A_7 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 6.1702 & -0.0041 & 0.6205 & 0.0001 \\ 0 & 0 & 0 & 1 \\ 8.8794 & -0.0193 & -1.0119 & 4.4 \times 10^{-5} \end{bmatrix}$$

$$A_8 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3.6421 & 0.0018 & 0.0721 & 0.0002 \\ 0 & 0 & 0 & 1 \\ 2.429 & -0.0305 & 2.9832 & -1.9 \times 10^{-5} \end{bmatrix}$$

$$A_9 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3.2933 & -0.0009 & -0.2188 & -1.2 \times 10^{-5} \\ 0 & 0 & 0 & 1 \\ -7.4649 & 0.0024 & 3.2693 & 9.2 \times 10^{-6} \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \\ -1 & 2 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, B_3 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 1 & 2 \end{bmatrix},$$

The triangle type of weighting functions for the triggering variables x_1 and x_3 in each rule are adopted for the T-S model as illustrated in Fig. 1 and $\lambda_k = 0.8$ for $k = 1, \dots, L$. The reference model (4) parameters are given by [11]

$$A_r = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -6 & -5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -6 & -5 \end{bmatrix}, B_r = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ and}$$

$$C_r = C_i.$$

Using Theorem 1 and the LMI optimization toolbox in Matlab, the following controller gains K_j for $j = 1, \dots, L$ are obtained

$$K_1 = \begin{bmatrix} -602.5242 & -266.3152 \\ -266.3152 & -649.2914 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} -547.3363 & -265.2308 \\ -265.2308 & -940.6244 \end{bmatrix},$$

$$K_3 = \begin{bmatrix} -486.4161 & -11.9060 \\ -11.9060 & -562.5966 \end{bmatrix},$$

$$K_4 = \begin{bmatrix} -670.0981 & -99.0489 \\ -99.0489 & -901.7503 \end{bmatrix},$$

$$K_5 = \begin{bmatrix} -782.7883 & -240.0344 \\ -240.0344 & -561.2339 \end{bmatrix},$$

$$K_6 = \begin{bmatrix} -634.0707 & -128.9711 \\ -128.9711 & -900.8147 \end{bmatrix},$$

$$K_7 = \begin{bmatrix} -438.1436 & 4.4070 \\ 4.4070 & -570.1235 \end{bmatrix},$$

$$K_8 = \begin{bmatrix} -542.5231 & -224.0683 \\ -224.0683 & -945.0132 \end{bmatrix},$$

$$K_9 = \begin{bmatrix} -614.0204 & -254.7631 \\ -254.7631 & -645.2517 \end{bmatrix}$$

Different reference and disturbance signals are applied in simulation of the closed loop system. Fig. 2-3 present the simulation results for the proposed tracking control for T-S model. The simulation results indicate that the closed loop system can efficiently track the reference model in the presence of external disturbances and the desired performance can be achieved. It is found that the tracking error between the reference output $y_r(t)$ and actual closed loop output $y(t)$ is always less than 0.03 and $|\dot{h}_k| \leq 0.4$ for $k = 1, \dots, L$.

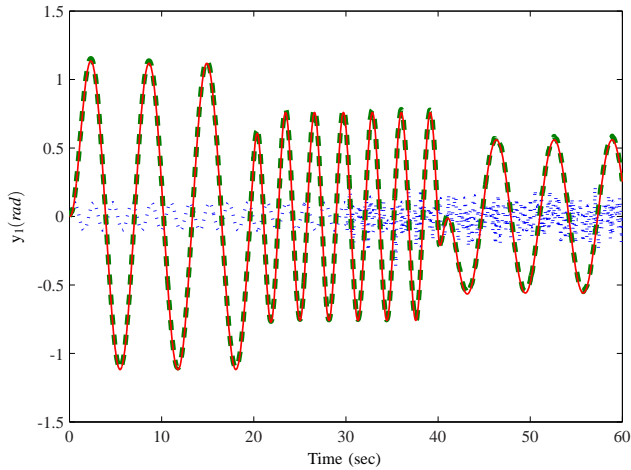


Fig. 2. The trajectories of the output variable y_1 (the proposed nonlinear control: dashed line), reference output y_{1r} (solid line) and external disturbance w_1 (dotted line)

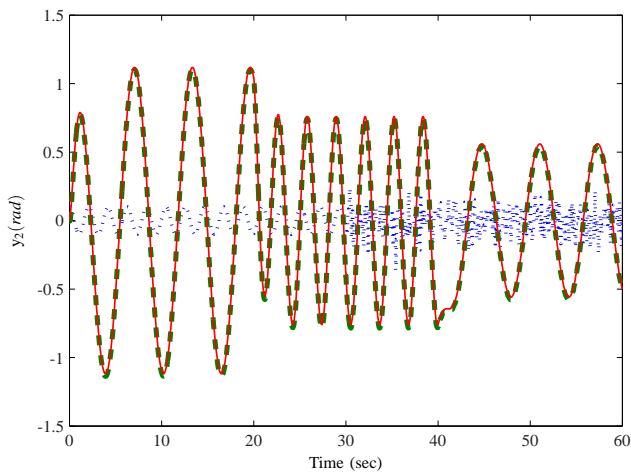


Fig. 3. The trajectories of the output variable y_2 (the proposed nonlinear control: dashed line), reference output y_{2r} (solid line) and external disturbance w_2 (dotted line)

VI. CONCLUSION

In this paper, the problem of reference model tracking of nonlinear systems has been considered. Based on the Takagi-Sugeno model, a nonlinear static output feedback controller is developed to reduce the tracking error by minimizing the attenuation level in terms of \mathcal{L}_2 gain. The novelty of the presented approach is that it is based on a parameter varying Lyapunov function. The advantage of the proposed tracking control design is that a simple static output feedback controller is used without feedback linearization technique and a complicated adaptive scheme. The complete synthesis problem has been formulated in terms of LMI feasibility which can be efficiently solved using a dedicated software. Simulation results based on a two-link robot arm have demonstrated the efficiency of the proposed algorithm.

REFERENCES

- [1] T. Takagi and M. Sugeno, "Fuzzy identification of systems and its applications to modeling and control", *IEEE trans. on Systems, Man, and Cybernetics*, vol. 15, no. 1, pp. 116-132, 1985.
- [2] L. Wang, "A course in fuzzy systems and control", Prentice Hall, London, UK, 1997.
- [3] K. Tanaka, and M. Sugeno, "Stability analysis and design of fuzzy control systems", *Fuzzy Sets Syst.*, vol. 45, no. 2, pp. 135-136, Jan. 1992.
- [4] S. Cao, N. Rees and G. Feng, "Analysis and design of fuzzy control systems using dynamic fuzzy global models", *Fuzzy Sets and Syst.*, vol. 75, no. 1, pp. 47-62, Oct. 1995.
- [5] L. Tanaka, and H. Wang, "Fuzzy control systems design and analysis: A linear matrix inequality approach", John Wiley & Sons, Inc., 2001.
- [6] K. Tanaka, T. Ikeada and H. Wang, "Fuzzy regulators and fuzzy observers: Relaxed stability conditions and LMI-based designs", *IEEE trans. on Fuzzy Systems*, vol. 6, no. 2, pp. 1-16, May 1998.
- [7] K. Tanaka, T. Hori and H. O. Wang, "A multiple Lyapunov function approach to stabilization of fuzzy systems" *IEEE Trans. on Fuzzy Systems*, vol. 11 no. 4, pp. 582-589, 2003.
- [8] B. Ding, H. Sun and P. Yang, "Further studies on LMI-based relaxed stabilization conditions for nonlinear systems in Takagi-Sugeno's form," *Automatica*, vol. 42, no. 3, pp. 503-508, Mar. 2006
- [9] M. Ezzeldin, H.M. Emar, A. Elshafei, and A. Bahgat, Observer based fuzzy-control design using relaxed LMI conditions, *IEEE Conference on Decision and Control*, pp. 5312 - 5317, Dec. 2007.
- [10] H. Fang, Y. Liu, W. Kau, L. Hong and H. Lee, "A new LMI-based approach to relaxed quadratic stabilization of T-S fuzzy control systems", *IEEE Trans. Fuzzy Syst.*, vol. 14, no. 3, pp. 386-397, Jun. 2006
- [11] C. S. Tseng, B. S. Chen, and H. J. Uang, Fuzzy tracking control design for nonlinear dynamic systems via T-S fuzzy model, *IEEE Trans. Fuzzy Syst.*, vol. 9, no. 3, pp. 381-392, Jun. 2001.
- [12] C. S. Tseng and B. S. Chen, H_∞ Decentralized fuzzy modeling reference tracking control design for nonlinear interconnected systems, *IEEE Trans. Fuzzy Syst.*, vol. 9, no. 6, pp. 795-809, Dec. 2001.
- [13] C. S. Tseng and B. S. Chen, Output Tracking Control for Fuzzy Via Output Feedback Design, *IEEE Trans. Fuzzy Syst.*, vol. 9, no. 6, pp. 795-809, Dec. 2001.
- [14] C.C. Kung and H.H. Li, Tracking control of non linear systems by fuzzy model-based controller, *Proc. IEEE Int. Conf.*, pp. 623-628, Jul. 1997.
- [15] H. Ying, Analytical analysis and feedback linearization tracking control of the general Takagi-Sugeno fuzzy dynamic systems, *IEEE Trans. Syst. Man, Cybern.*, pp. 290-298, May 1999
- [16] Landua and D. Youan. "Adaptive Control: The Model Reference Approach," New York: Dekker, 1979.
- [17] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, "Linear Matrix Inequalities in System and Control Theory," Philadelphia, (PA:SIAM), 1994.
- [18] Stephen Boyd and Lieven Vandenberghe, "Convex Optimization" 2004, (Cambridge University Press), Internet site, <http://www.cambridge.org>.
- [19] H.D.Tuan, P. Apkarian, T.NariKiyoo and Y.Yamamoto "Parameterized linear matrix inequality techniques in fuzzy system design," *IEEE Trans. Fuzzy Syst.*, vol. 9, no.2, pp. 324-332, Apr. 2001.