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Brief paper

Spectral analysis of block structured nonlinear systems and higher order sinusoidal input describing functions[☆]

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ABSTRACT

When analyzing and modeling dynamical systems in the frequency domain, the effects of nonlinearities need to be taken into account. This paper contributes to the analysis of the effects of nonlinearities in the frequency domain by supplying new analytical tools and results that allow spectral analysis of the output of a class of nonlinear systems. A mapping from the parameters defining the nonlinear and LTI dynamics to the output spectrum is derived, which allows analytic description and analysis of the corresponding higher order sinusoidal input describing functions. The theoretical results are illustrated by examples that show both the use and efficiency of the proposed algorithms.

1. Introduction

Dynamical systems are often modeled and analyzed in the frequency domain by identifying their frequency response function (FRF). Modeling systems in the frequency domain, rather than the time domain, offers important advantages such as easy interpretation of the system dynamics and has led to significant progress in linear controller design (Bode, 1945). However, when nonlinearities are present, the FRF fails to model the complete system dynamics. Use of such models without taking nonlinearities into account can lead to unexpected and undesired results. Amplitude dependent gains can for example render a designed controller unstable and harmonics generated by nonlinearities may excite unwanted dynamics. Therefore, when extending frequency domain techniques to nonlinear systems, the effects of nonlinearities in the frequency domain have to be taken into account. Numerous studies have been performed to investigate the frequency domain modeling of nonlinear systems. Next, four approaches relevant to the presented work are discussed.

Generalized FRF The Generalized Frequency Response Function (GFRF) provides a generalization of the FRF for linear time invariant systems to nonlinear systems. The GFRF was first introduced in George (1959) and methods to measure and interpret the GFRF were developed in Billings and Tsang (1989). This research is continued in for example Li and Billings (2011) and in Yue, Billings, and Lang (2005) where the GFRFs are interpreted in terms of multidimensional frequency spaces and visualization techniques are developed. In Shanmugam and Jong (1975) the GFRFs are related to the parameters of block structured nonlinear systems similar to the systems analyzed in this paper. Application of the GFRFs to force a required output spectrum from a nonlinear system is presented in Jing, Lang, and Billings (2010) (see also Nuij, Steinbuch, and Bosgra (2008c)). Furthermore, the analysis of the effects of time domain model parameters on the frequency domain behavior of the nonlinear system (Jing, Lang, Billings, & Tomlinson, 2006) led to the definition of the output frequency response function (OFRF) in Lang, Billings, Yue, and Li (2007).

FRF for nonlinear systems In Pavlov, Van de Wouw, and Nijmeijer (2007) the notion of an FRF for uniformly convergent nonlinear systems (Pavlov, Pogromsky, Van De Wouw, & Nijmeijer, 2004) is introduced. This function maps a harmonic input signal to the output of the system by means of a state or output FRF. The authors also propose an extension of the concept of the Bode plot,

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to visualize part of the nonlinear dynamics. These ideas lead to a well defined notion of performance for a class of (controlled) nonlinear systems as demonstrated in Van de Wouw, Pastink, Heertjes, Pavlov, and Nijmeijer (2008) in an application to variable gain control design for optical storage drives.

BLA and detection of nonlinearities When identifying frequency domain models, the quality of the model with respect to linearity needs to be assessed. In Pintelon and Schoukens (2001) frequency domain identification methods are discussed which provide a quantitative measure for the level and type of nonlinear influences. The authors utilize the properties of a special class of multisine excitation signals (Pintelon, Vandersteen, De Locht, Rolain, & Schoukens, 2004) and averaging techniques to derive a best linear approximation (BLA) (Schoukens, Pintelon, Dobrowiecki, & Rolain, 2005; Schoukens, Lataire, Pintelon, Vandersteen, & Dobrowiecki, 2009) of the systems dynamics and to quantify nonlinear influences.

HOSIDF The Higher Order Sinusoidal Input Describing Functions (HOSIDFs) are introduced in Nuij, Bosgra, and Steinbuch (2006) and are an extension of the Sinusoidal Input Describing Function (SIDF) (Gelb & Vander Velde, 1968). Other than the SIDF, the HOSIDFs describe the systems response (gain and phase) to a sinusoidal input signal, at harmonics of the excitation frequency. Open and closed loop identification of the HOSIDFs are discussed in Nuij et al. (2006) and Nuij, Steinbuch, and Bosgra (2008a). The HOSIDFs are compared to the best linear approximation in Rijlaarsdam, van Loon, Nuij, and Steinbuch (2010) and used to derive physical (friction) parameters in Nuij, Steinbuch, and Bosgra (2008b). Finally, the application of HOSIDFs to (nonlinear) controller design for nonlinear systems yields significant advantages over conventional time domain tuning (Rijlaarsdam, Nuij, Schoukens, & Steinbuch, 2011).

Contribution This paper contributes to the analysis of the effects of nonlinearities in the frequency domain by supplying new analytical tools and results that allow for the analysis of the output spectra of nonlinear systems and the corresponding Higher Order Sinusoidal Input Describing Functions (HOSIDFs). Compared to the GFRF, the 'FRF for nonlinear systems' and the BLA, the main disadvantage of the results presented in this paper is the limited class of excitation signal considered. Although the class of systems considered is significant, the methods presented in this paper are applicable for sinusoidal inputs only. However, the response of a nonlinear system to a sinusoidal input signal already yields valuable information in, for example, optimal nonlinear control design (Rijlaarsdam et al., 2011).

The main advantage of the results presented in the following, compared to the GFRF, is their simplicity and intuitive insight in the effects of nonlinearities in the frequency domain. This yields a clear extension to identification purposes as recently demonstrated in Rijlaarsdam (2011) where novel techniques for broadband identification of the HOSIDFs are presented. An important advantage of the results presented here over the FRF for nonlinear systems and the BLA is that they provide detailed phase information about nonlinear influences. This is found to be crucial in control applications (Rijlaarsdam et al., 2011).

Structure This paper deals with the spectral analysis of systems with polynomial nonlinearities. First, in Section 3, an analytical expression for the output spectra of a class of block structured nonlinear systems is derived. Next, in Section 4, the Higher Order Sinusoidal Input Describing Functions (HOSIDFs) are introduced and analytical expressions for the HOSIDFs of a class of nonlinear systems are derived. Finally, Section 5 presents illustrative examples. Matlab tools to apply the presented results are available online.[2]

2. Nomenclature and preliminaries

In the following, the continuous time signals considered are denoted by non-capitalized roman letters $x(t) \in \mathbb{R}$. The corresponding continuous spectra $\mathscr{X}(\omega) \in \mathbb{C}$ are denoted in capitalized, calligraphic font. Unless specified otherwise, single sided spectra are considered. Frequent use is made of vectors containing only specific spectral components $X[\ell] = \mathscr{X}\big((\ell-1)\omega_0\big)$ denoted in capitalized roman letters. Hence, $X[\ell] \in \mathbb{C}$ contains the spectral component at the $k = (\ell-1)$th harmonic of the excitation frequency ω_0. Finally, matrices are denoted by capitalized Greek characters.

3. Spectral analysis of nonlinear systems

In this section new efficient analytical results are introduced which allow us to model the input–output behavior of a class of nonlinear systems with polynomial nonlinearities. These results are applied in the analysis of block structured nonlinear systems and allow frequency domain analysis of such systems by means of the higher order sinusoidal input describing functions.

3.1. Spectral analysis of polynomial nonlinearities

Consider a static polynomial nonlinearity of degree P:

$$y(t) = \sum_{p=1}^{P} \alpha_p u^p(t), \tag{1}$$

with $u(t), y(t) \in \mathbb{R}$ the input and output of the system and $\alpha_p \in \mathbb{R}$ the polynomial coefficients. Next, consider the output spectrum $\mathscr{Y}(\omega)$ when system (1) is subject to a one-tone input:

$$u(t) = \gamma \cos(\omega_0 t + \varphi_0), \tag{2}$$

with $\gamma, \varphi_0 \in \mathbb{R}$ the amplitude and phase and $\omega_0 \in \mathbb{R}_{>0}$ the frequency of the input signal. If (1) is subject to (2) the output spectrum $\mathscr{Y}(\omega) \in \mathbb{C}$ consists solely of harmonics $k\omega_0, k = 0, 1, 2, \ldots$ of the input frequency, i.e.

$$\begin{aligned} y(t) &= \sum_{p=1}^{P} \alpha_p \gamma^p \cos^p(\omega_0 t + \varphi_0) \\ &= \sum_{k=0}^{P} A_k(\gamma) \cos\big(k\omega_0 t + \phi_k(\varphi_0)\big) \end{aligned} \tag{3}$$

with $A_k(\gamma), \phi_k(\varphi_0) \in \mathbb{R}$ the amplitude and phase of the kth harmonic component. Next, the power series in (3) is rewritten to the sum of one tones using the following relationship (Gradshteyn & Ryzhik, 2000):

$$\begin{aligned} \cos^{2n-\sigma}(x) = \frac{1}{2^{2n-1-\sigma}} \Bigg\{ &(1-\sigma)\binom{2n}{n} \\ &+ \sum_{m=0}^{n-1} \binom{2n-\sigma}{m} \cos\left((2(n-m)-\sigma)x\right) \Bigg\}, \end{aligned} \tag{4}$$

with $n \in \mathbb{N}_{\geq 1}$, $\sigma \in \{0, 1\}$ and the binomial coefficient $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ $\forall\ a, b \in \mathbb{N}, 0 \leq b \leq a$ and 0 otherwise. Applying (4) to each of the terms on the left hand side of (3) and taking the Fourier transform[3] yields the corresponding output spectrum $\mathscr{Y}(\omega)$ of (1):

[2] www.davidrijlaarsdam.nl.

[3] $\mathrm{FT}\{x(t)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t)e^{i\omega t}\mathrm{d}t$.

$$\mathrm{FT}\{\alpha_p\gamma^p\cos^p(\cdot)\} = \alpha_p\gamma^p(1-\sigma_p)\frac{n}{2^{2n}}\binom{2n}{n}\delta(0) + \frac{\alpha_p\gamma^p}{2^{2n-\sigma_p}}\sum_{m=0}^{n-1}\binom{2n-\sigma_p}{m} \times \left(c\delta(\omega-d\omega_0)+c^\star\delta(\omega+d\omega_0)\right), \quad (5)$$

with $\delta(\cdot)$ the Dirac delta function, $\sigma_p = p \bmod 2$, $n = \frac{1}{2}(p+\sigma_p)$, $d = 2(n-m)-\sigma_p \in \mathbb{N}_{\geq 1}$, $c = \mathrm{e}^{\mathrm{i}d\varphi_0} \in \mathbb{C}$ and * the complex conjugate. Next, consider the contribution of all polynomial terms to a spectral component $\mathscr{Y}(k\omega_0)$ by considering terms in (5) such that $d = 2(n-m)-\sigma_p = k$, for each power p in (1). Summing these contributions over all p yields the spectral components of the single sided spectrum at harmonics $k\omega_0$ of the input frequency:

$$\mathscr{Y}(0) = \sum_{p=1}^{P}(1-\sigma_p)\left[\alpha_p\left(\frac{\gamma}{2}\right)^p\binom{p}{\frac{p}{2}}\frac{p}{2}\right] \quad (6)$$

$$\mathscr{Y}(k\omega_0) = \sum_{p=1}^{P}\sigma_{pk}\left[2\alpha_p\left(\frac{\gamma}{2}\right)^p\binom{p}{\frac{p-k}{2}}\mathrm{e}^{\mathrm{i}k\varphi_0}\right] \quad (7)$$

with $k \in \mathbb{N}_{\geq 1}$ and $\sigma_{p(k)}$ defined in Theorem 1. This yields a generalized relation that is formalized in Theorem 1.

Theorem 1 (*Nonlinear Coeff. and Output Spectra*)**.** *Consider a static polynomial mapping (1), subject to a one-tone input (2). Then, the single sided spectrum of the output $y(t)$ is given by the following mapping $\mathbb{R}^P \mapsto \mathbb{C}^{P+1}$, from the polynomial coefficients $\boldsymbol{\alpha}$ to the output spectrum $\mathscr{Y}(\omega)$:*

$$Y = \Phi(\varphi_0)\ \Omega\ \Gamma(\gamma)\boldsymbol{\alpha}, \quad (8)$$

where the different components are defined below.

Output spectrum (vector) $Y \in \mathbb{C}^{P+1}$
Where $Y = [\mathscr{Y}(0)\ \mathscr{Y}(\omega_0)\ \mathscr{Y}(2\omega_0)\ \cdots\ \mathscr{Y}(P\omega_0)]^T$ is a vector containing the nonzero spectral lines in the output spectrum, at harmonics of the input frequency.

Input phase matrix $\Phi(\varphi_0) \in \mathbb{C}^{(P+1)\times(P+1)}$
Describing the influence of the input phase on the output spectrum: $\Phi_{k+1,k+1}(\varphi_0) = \mathrm{e}^{\mathrm{i}k\varphi_0}$, $k = 0, 1, 2, \ldots$ and 0 otherwise.

Input gain matrix $\Gamma(\gamma) \in \mathbb{R}^{P\times P}$
Describing the influence of the input amplitude on the output spectrum: $\Gamma_{p,p}(\gamma) = \left(\frac{\gamma}{2}\right)^p$ and 0 otherwise.

Inter-harmonic gain matrix $\Omega \in \mathbb{R}^{(P+1)\times P}$
Describing the relation between the input and the harmonic components in the output spectrum:

$$\Omega_{1,p} = (1-\sigma_p)\binom{p}{\frac{p}{2}}\frac{p}{2}$$

$$\Omega_{k+1,p} = 2\binom{p}{\frac{p-k}{2}}\sigma_{pk} \quad \forall k \leq p, k \in \mathbb{N}_{\geq 1}$$

and 0 otherwise. With $\sigma_p = p \bmod 2$, $\sigma_k = k \bmod 2$ and $\sigma_{pk} = \sigma_p\sigma_k + (1-\sigma_p)(1-\sigma_k)$.

Polynomial coefficients $\boldsymbol{\alpha} \in \mathbb{R}^P$
Where $\boldsymbol{\alpha} = [\alpha_1\ \alpha_2\ \cdots\ \alpha_P]^T$ is a vector containing the coefficients of the polynomial nonlinearity.

Proof. (8) follows from (6) to (7) by rewriting the sums to a matrix product. □

Remark. Eq. (8) can be interpreted as an OFRF (Lang et al., 2007) for static polynomial systems.

Theorem 1 allows numerically efficient computation of output spectra of a class of nonlinear systems and provides insight into the mechanisms that generate these spectra. Next, these results are extended to a general class of block structured nonlinear dynamical systems.

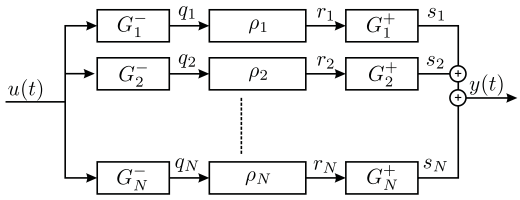

Fig. 1. $\overline{\mathbb{LPL}}$ block structured system.

3.2. Analysis of block structured nonlinear systems

This section presents the analysis of a class of nonlinear systems called parallel Wiener–Hammerstein systems with polynomial nonlinearities (Giri & Bai, 2010; Schoukens, Pintelon, & Rolain, 2011), which are defined below.

Definition 1 ($\overline{\mathbb{LPL}}$*: Block Structures*)**.** Consider an N-branch, block structured configuration as depicted in Fig. 1. Each branch consists of a series connection of a LTI block $G_n^-(\omega)$, a static nonlinear mapping ρ_n and another LTI block $G_n^+(\omega)$. The system has one input $u(t)$, one output $y(t)$ and intermediate signals $q_n(t)$, $r_n(t)$ and $s_n(t)$. The nonlinearity $\rho_n: \mathbb{R} \mapsto \mathbb{R}$ is a static, polynomial mapping of degree P_n:

$$\rho_n : r_n(t) = \sum_{p=1}^{P_n}\alpha_p^{[n]}q_n^p(t) \quad (9)$$

with $\alpha_p^{[n]} \in \mathbb{R}$. If $G_n^-(\omega) = 1$ or $G_n^+(\omega) = 1 \forall n \in \mathbb{N}_{\geq 1}$, the remaining $\overline{\mathbb{PL}}$ or $\overline{\mathbb{LP}}$ system equals a parallel Hammerstein or Wiener system with polynomial nonlinearities.

The output spectrum of $\overline{\mathbb{LPL}}$ systems, subject to (2), follows from Theorem 1. Consider the input $u(t)$ as it passes through the first linear block $G_n^-(\omega)$. The filtered signal $q_n(t)$ then serves as an input for the nonlinear block ρ_n, which yields the corresponding output spectrum R_n.

$$R_n(\omega_0, \varphi_0, G_n^-) = \Phi\left(\varphi_0 + \angle G_n^-(\omega_0)\right)\ \Omega\ \Gamma\left(\gamma|G_n^-(\omega_0)|\right)\boldsymbol{\alpha}^{[n]}.$$

Next, each signal $r_n(t)$ is modified by the corresponding (second) linear block $G_n^+(\omega)$, which yields the output spectrum of the nth branch

$$\mathscr{S}_n(\omega_0, \varphi_0, G_n^-, G_n^+) = \Delta(\omega_0)G_n^+(\omega)\Phi\left(\varphi_0 + \angle G_n^-(\omega_0)\right)\ \Omega\ \Gamma\left(\gamma|G_n^-(\omega_0)|\right)\boldsymbol{\alpha}^{[n]} \quad (10)$$

where $\Delta(\omega_0)$ is defined in Lemma 1 yielding an analytic description of the output spectrum of a $\overline{\mathbb{LPL}}$ system.

Lemma 1 (*Spectrum of $\overline{\mathbb{LPL}}$ Systems*)**.** *Consider an $\overline{\mathbb{LPL}}$ system subject to a one-tone input (2). Then, the output spectrum equals:*

$$Y(\omega_0, \gamma, \varphi_0, G_n^{\pm}) = \sum_{n=1}^{N}\Delta(\omega_0)G_n^+(\omega)\ \Phi(\psi_n)\ \Omega\ \Gamma(\gamma_n)\boldsymbol{\alpha}^{[n]} \quad (11)$$

with $\gamma_n = \gamma|G_n^-(\omega_0)|$, $\psi_n = \varphi_0 + \angle G_n^-(\omega_0)$. Moreover, $\Delta(\omega_0) = \mathrm{diag}([\delta(\omega-0)\ \ \delta(\omega-\omega_0)\ \ \delta(\omega-2\omega_0)\ \ \cdots\ \ \delta(\omega-P_n\omega_0)]) \in \mathbb{R}^{(P_n+1)\times(P_n+1)}$ is a diagonal matrix of δ-functions, such that: $\Delta(\omega_0)G_n(\omega) = \mathrm{diag}([G(0)\ G(\omega_0)\ G(2\omega_0)\ \cdots\ G(P_n\omega_0)])$.

Proof. (11) follows from summation of (10) over all N branches. □

Lemma 1 extends the results in Theorem 1 to $\overline{\mathbb{LPL}}$ systems. Next, these results are used to derive analytical expressions for the corresponding HOSIDFs.

4. Higher Order Sinusoidal Input Describing Functions

The Higher Order Sinusoidal Input Describing Functions (HOSIDFs) are introduced in Nuij et al. (2006). In this section a new definition of the HOSIDF is provided and the HOSIDFs of $\overline{\mathbb{LPL}}$ systems are analyzed.

4.1. Definition and analysis

In Nuij et al. (2006) the output of a class of nonlinear systems, subject to (2) is considered. This output is composed of K harmonics of the input frequency, i.e.

$$y(t)=\sum_{k=0}^{K}|\mathfrak{H}_k(\omega_0,\gamma)|\gamma^k\cos\left(k(\omega_0 t+\varphi_0)+\angle\mathfrak{H}_k(\omega_0,\gamma)\right),$$

where $\mathfrak{H}_k(\omega_0,\gamma)\in\mathbb{C}$ is the kth order HOSIDF which describes the response (gain and phase) at harmonics of the excitation frequency ω_0.

Definition 2 ($\mathfrak{H}_k(\omega,\gamma)$*: HOSIDF*)**.** Consider a uniformly convergent, time invariant nonlinear system (Pavlov et al., 2004) subject to (2). Define the system's steady state output $y(t)$ and single sided spectra of the input and output $\mathscr{U}(\omega)$, $\mathscr{Y}(\omega)\in\mathbb{C}$. Then, the kth Higher Order Sinusoidal Input Describing Function $\mathfrak{H}_k(\omega_0,\gamma)\in\mathbb{C}$, $k=0,1,2,\ldots$ is defined as:

$$\mathfrak{H}_k(\omega_0,\gamma)=\frac{\mathscr{Y}(k\omega_0)}{\mathscr{U}^k(\omega_0)}. \tag{12}$$

Remark. In (12) the amplitude dependency $|\mathfrak{F}_k|\propto\gamma^k$ inherent to the original HOSIDF model structure is removed. Hence, it reveals only system characteristics.

4.2. HOSIDFs of block structured systems

Using Definition 2 and Lemma 1 the corresponding HOSIDFs of $\overline{\mathbb{LPL}}$ systems are defined in Lemma 2.

Lemma 2 (*HOSIDFs of* $\overline{\mathbb{LPL}}$ *Systems*)**.** *The HOSIDFs of a N-branch $\overline{\mathbb{LPL}}$ system equal*

$$H(\omega_0,\gamma,G_n^{\pm})=\Upsilon^{-1}\sum_{n=1}^{N}\Delta(\omega_0)\,G_n^{+}(\omega)\left[\Phi(\angle G_n^{-}(\omega_0))\Omega\,\Gamma\right.$$
$$\left.\times(|G_n^{-}(\omega_0)|\gamma)\boldsymbol{\alpha}^{[n]}\right] \tag{13}$$

with $H=[\mathfrak{H}_0(\omega_0)\ \mathfrak{H}_1(\omega_0)\ \mathfrak{H}_2(\omega_0)\cdots\mathfrak{H}_{\max_n P_n}(\omega_0)]^T$ and the gain compensation matrix $\Upsilon_{k+1,k+1}(\gamma)=\gamma^k$ and 0 otherwise. (For proof see Appendix A.)

Remark. The results in Lemma 2 show that the HOSIDFs of $\overline{\mathbb{LPL}}$ systems are independent of the input phase.

Moreover, for $\overline{\mathbb{PL}}$ systems, a set of excitation amplitude independent basis functions for the HOSIDFs exists.

Definition 3 ($\mathfrak{F}_p(\omega)$*: fHOSIDFs of* $\overline{\mathbb{PL}}$ *Systems*)**.** The Fundamental Higher Order Sinusoidal Input Describing functions (fHOSIDFs) $\mathfrak{F}_p(\omega)\in\mathbb{C}$ of a $\overline{\mathbb{PL}}$ system equal a linear combination of the LTI dynamics $G_n^{+}(\omega)$ such that ρ_n: $r_n(t)=q_n^n$, i.e.

$$\mathfrak{F}_p(\omega_0)=\sum_{n=1}^{N}G_n^{+}(\omega_0)\alpha_p^{[n]}. \tag{14}$$

Remark. The fHOSIDFs are amplitude independent basis functions for the HOSIDFs and provide a decoupling of amplitude and frequency effects in the HOSIDFs, since $H(\omega_0,\gamma,G_n^{+})=\left[\Upsilon^{-1}(\gamma)\Delta(\omega_0)\,\Omega\,\Gamma(\gamma)\right]F(\omega_0)$, with $F(\omega)=[\mathfrak{F}_1(\omega_0)\ \ \mathfrak{F}_2(\omega_0)\cdots\mathfrak{F}_{\max_n P_n}(\omega_0)]^T$.

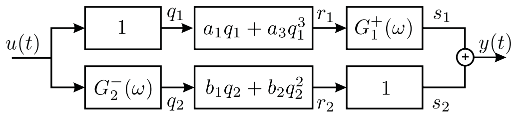

Fig. 2. Two-branch $\overline{\mathbb{LPL}}$ system.

5. Examples

The following examples illustrate the presented results.

Example 1 (*Output Spectrum*)**.** Consider a polynomial mapping (1) with $P=3$ subject to (2). Then, the the output spectrum is readily computed using Theorem 1.

$$Y=\Phi(\varphi_0)\ \Omega\ \Gamma(\gamma)\boldsymbol{\alpha}$$
$$=\begin{bmatrix}1&0&0&0\\0&e^{1i\varphi_0}&0&0\\0&0&e^{2i\varphi_0}&0\\0&0&0&e^{3i\varphi_0}\end{bmatrix}\begin{bmatrix}0&2&0\\2&0&6\\0&2&0\\0&0&2\end{bmatrix}$$
$$\times\begin{bmatrix}\left(\frac{\gamma}{2}\right)^1&0&0\\0&\left(\frac{\gamma}{2}\right)^2&0\\0&0&\left(\frac{\gamma}{2}\right)^3\end{bmatrix}\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix}.$$

Example 2 (*Spectrum & HOSIDFs of* $\overline{\mathbb{LPL}}$ *Systems*)**.** Consider the two branch $\overline{\mathbb{LPL}}$ system depicted in Fig. 2, subject to (2). Application of Lemma 1 immediately yields an analytic expression for the output spectrum:

$$Y=\Delta(\omega_0)G_1^{+}(\omega)\Phi(\varphi_0)\ \Omega\ \Gamma(\gamma)\boldsymbol{\alpha}^{[1]}$$
$$+\Delta(\omega_0)\Phi(\varphi_0+\angle G_2^{-})\ \Omega\ \Gamma(\gamma|G_2^{-}|)\boldsymbol{\alpha}^{[2]}$$
$$=\begin{bmatrix}\frac{b_2\gamma^2}{2}|G_2^{-}(\omega_0)|^2\\ \gamma\left[\left(a_1+\frac{3}{4}a_3\gamma^2\right)G_1^{+}(\omega_0)+b_1G_2^{-}(\omega_0)\right]e^{i\varphi_0}\\ \frac{b_2\gamma^2}{2}\left[G_2^{-}(\omega_0)\right]^2e^{2i\varphi_0}\\ \frac{a_3\gamma^3}{4}G_1^{+}(3\omega_0)e^{3i\varphi_0}\end{bmatrix}.$$

The corresponding HOSIDFs are computed using Lemma 2 and require neither computation of the output spectrum, nor knowledge of the phase of the input signal, i.e.

$$H=\begin{bmatrix}\mathfrak{H}_0(\omega_0)\\\mathfrak{H}_1(\omega_0)\\\mathfrak{H}_2(\omega_0)\\\mathfrak{H}_3(\omega_0)\end{bmatrix}$$
$$=\begin{bmatrix}\frac{b_2\gamma^2}{2}|G_2^{-}(\omega_0)|^2\\ \left[\left(a_1+\frac{3}{4}a_3\gamma^2\right)G_1^{+}(\omega_0)+b_1G_2^{-}(\omega_0)\right]\\ \frac{b_2}{2}\left(G_2^{-}(\omega_0)\right)^2\\ \frac{a_3}{4}G_1^{+}(3\omega_0)\end{bmatrix}.$$

6. Conclusion

This paper presents new, efficient analytical tools and results that allow spectral analysis of the output of a class of nonlinear systems. This provides insight into the dynamics of block structured dynamical systems and allows analytic description and analysis of the corresponding Higher Order Sinusoidal Input Describing Functions (HOSIDFs). Given the systems linear dynamics, the output spectra and HOSIDFs can be described as a simple polynomial function of the parameters defining the nonlinearity.

Appendix. Proof of Lemma 2

For deterministic spectra $\mathscr{U}(\omega)$, $\mathscr{Y}(\omega)$ and input (2), Eq. (12) yields: $\mathfrak{H}_k(\omega_0, \gamma) = \frac{e^{-ik\varphi_0}\mathscr{Y}(k\omega_0)}{|\mathscr{U}(\omega_0)|^k}$. Hence $H = \Upsilon^{-1}(\gamma)\Phi^{-1}(\varphi_0)Y$. Using (11) yields: $H(\omega_0, G_n^{\pm}) = \Upsilon^{-1}\Phi^{-1}(\varphi_0)\sum_{n=1}^{N}\Delta(\omega_0) G_n^{+}(\omega)\,\Phi(\psi_n)\,\Omega\,\Gamma(\gamma_n)\boldsymbol{\alpha}^{[n]}$, with $\gamma_n = \gamma|G_n^{-}(\omega_0)|$, $\psi_n = \varphi_0 + \angle G_n^{-}(\omega_0)$. Finally, $\Phi(\cdot)$, Δ are diagonal matrices and $\Phi^{-1}(\varphi_0)\Phi(\varphi_0 + \angle G_n^{-}(\omega_0)) = \Phi(\angle G_n^{-}(\omega_0))$, which yields (13). □

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